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Isotopy invariants for smooth tori in 4-manifolds

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Abstract

In this paper we define invariants under smooth isotopy for certain two-dimensional knots using some refined Cerf theory. One of the invariants is the knot type of some classical knot generalizing the string number of closed braids. The other invariant is a generalization of the unique invariant of degree 1 for classical knots in 3-manifolds. Possibly, these invariants can be used to distinguish smooth embeddings of tori in some 4-manifolds but which are equivalent as topological embeddings.

Keywords: Isotopy invariants; Smooth surfaces in 4-mainfolds; Two-dimensional braids

1. Introduction and announcement of the results

Let K denote an oriented (tame) knot in $F^2 \times \mathbb{R}$, where F^2 is any orientable connected and non-simply connected surface. Let $[K]$ denote the homology class represented by K in $H_1(F^2; \mathbb{Z})$.

In [4] we introduced the invariant W_K. It is an invariant of isotopy for $K \hookrightarrow F^2 \times \mathbb{R}$ and it is of degree 1 in the sense of the theory of finite-type invariants. It can be interpreted as an application

$$W_K : H_1(F^2; \mathbb{Z}) \backslash \{0, [K]\} \to H_0(F^2 \times \mathbb{R}; \mathbb{Z}) \cong H_0(F^2; \mathbb{Z}) \cong \mathbb{Z}$$

(cf. Section 2).

In [5] the invariant W_K was generalized to invariants of higher degrees and even of infinite degree and the surface F^2 was allowed to be non-orientable as well.

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Let us consider the particular case of a closed braid $K \hookrightarrow (S^1 \times \mathbb{R}) \times \mathbb{R}$ (see e.g. [1]). It follows from homological considerations that the string number of K, which can be interpreted as the knot type of the fiber $K \cap (\{s\} \times \mathbb{R}) \times \mathbb{R} \hookrightarrow \{s\} \times \mathbb{R}^2$ for any fixed $s \in S^1$, is invariant under *all* isotopies from a closed braid to a closed braid in the solid torus $S^1 \times \mathbb{R}^2$.

In this paper we generalize both the knot type of the fiber and the invariant W_K to invariants of knots in one dimension higher.

Viro introduced the notion of two-dimensional (closed) braids in $S^2 \times \mathbb{R}^2$ (see e.g. [9]). A smoothly embedded closed surface in $S^2 \times \mathbb{R}^2$ is a closed braid if the restriction on the surface of the natural projection onto S^2 is a ramified covering. In this definition the fiber is the same as in the case of classical closed braids but the projection is on S^2 instead of S^1. The advantage of this definition is its generality: Viro has shown that the analogue of Alexander's theorem is true for closed two-dimensional braids too. The disadvantage is, that it seems to be difficult to use parametrized Morse theory in order to study these two-dimensional braids.

In this paper we study another and much more special generalization of closed braids, which goes back to the notion of "motions of links in 3-manifolds" first studied by D. Goldsmith. In this definition the dimension of the fibers increases but the projection is still onto S^1. But distinct from D. Goldsmith we adapt a point of view from four-dimensional topology.

Let F^2 be an arbitrary orientable connected and non-simply connected surface and let T denote a smoothly embedded torus in the 4-manifold $M^4 = F^2 \times \mathbb{R} \times S^1$.

Let $\arg: M^4 \to S^1$ denote the natural projection on the S^1-factor.

Definition 1.1. T is called a *two-dimensional* (*closed*) *braid* if the restriction arg: $T \to S^1$ is a submersion.

Remark. (1) Our definition evidently implies that the orientable surface T has to be a torus.

(2) If we replace the surface F^2 by $\mathbb{R}$ and the torus T by a union of circles then we obtain the usual definition of a closed braid.

Let $T \hookrightarrow M^4$ be a two-dimensional braid. We assume that T, $F^2, \mathbb{R}, S^1$ and, hence, M^4 are oriented. Let T_S denote the intersection of T with the fiber $\arg^{-1}(s)$ for a fixed $s \in S^1$. Consequently, T_S is an oriented link in the oriented 3-manifold $\arg^{-1}(s) \cong F^2 \times \mathbb{R} \times \{s\} \cong F^2 \times \mathbb{R}$.

We assume in the following that T_S is a knot, i.e. connected, and that T_S is not contained in any 3-ball D^3 in $F^2 \times \mathbb{R}$.

It is well known that the band connected sum of a classical knot K with an unknot can change the knot type of K. The basic observation in this paper is that this cannot happen if the band connected sum with $K(= T_S)$ is induced by a smooth isotopy of a torus in $F^2 \times \mathbb{R} \times S^1$ *which starts from a two-dimensional braid*. As a corollary we obtain that the knot type of $T_S \hookrightarrow F^2 \times \mathbb{R}$ is invariant under *all* smooth isotopies which connect two-dimensional braids (Theorem 5.1).

Remark. A theorem of Artin [2] says that two closed braids which represent isotopic knots in the solid torus are, in fact, isotopic through closed braids. Of course, Theorem 5.1 would follow from the analogue of Artin's theorem for two-dimensional braids. However, we do not know whether or not such an analogue is true.

Next, we construct an invariant, called W_T, for two-dimensional braids and which can even distinguish two-dimensional braids with the same knot type $T_S \hookrightarrow F^2 \times \mathbb{R}$. We prove that W_T is invariant not only under isotopies of T which stay within the category of two-dimensional braids but *all* smooth isotopies of T in M^4 (Theorem 6.1).

The invariant can be interpreted as an application

$$W_T : H_1(F^2;\mathbb{Z})\backslash\{0,[T_S]\} \to H_1(M^4;\mathbb{Z}) \cong H_1(F^2;\mathbb{Z}) \oplus H_1(S^1;\mathbb{Z}).$$

It has the property that the restriction

$$W_T : H_1(F^2;\mathbb{Z})\backslash\{0,[T_S]\} \to H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$$

coincides with the invariant W_{T_s} introduced before.

The paper is organized as follows:

In Section 2 for completness we recall the definition of the invariant $W_K = W_{T_s}$. In Section 3 we construct the invariant W_T. In Section 4 we show its invariance under isotopies within the category of two-dimensional braids. In Section 5 we show that the knot type of the fiber T_S in $F^2 \times \mathbb{R}$ is invariant under all smooth isotopies which connect two-dimensional braids. In Section 6 we show that W_T is invariant under all smooth isotopies connecting two-dimensional braids. In Section 7 we calculate the invariants in some examples. In particular, we give lots of tori in $S^1 \times S^1 \times \mathbb{R}^2$ which are not smoothly isotopic but which are regularly homotopic and have diffeomorphic complements.

2. The invariant W_K for knots in 3-manifolds $F^2 \times \mathbb{R}$

We assume that the knot K is in general position with respect to the projection $F^2 \times \mathbb{R} \to F^2$. In this case we can look at K as a *diagram* on F^2. Let p be a crossing of this diagram of K (Fig. 1). We associate to p its writhe $w(p) = \pm 1$ in the usual way. Splitting the crossing p with respect to the orientation of K we obtain two oriented diagrams. Those of these diagrams which contain the undercross in p which go to the overcross in p, is denoted by K_p^+:

The *type* of the crossing p is defined as the homology class $[K_p^+] \in H_1(F^2;\mathbb{Z})$.

We define W_K as an element of the group ring $\mathbb{Z}[H_1(F^2;\mathbb{Z})]$.

Definition 2.1.

$$W_K = \sum_p w(p)[K_p^+], \quad [K_p^+] \notin \{0,[K]\}.$$

It is not difficult to show that W_K is an isotopy invariant of degree 1 (cf. [4,5, Section 2.2]; [11] for the most general definition of W_K).

Remark. Notice that Reidemeister moves of type I (see Fig. 2) create only crossings p of type 0 or $[K]$. Moreover, notice that in order to define W_K we need the orientation on the $\mathbb{R}$-factor of $F^2 \times \mathbb{R}$.

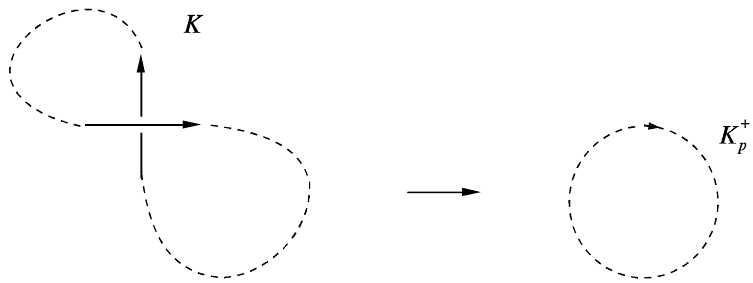

Fig. 1.

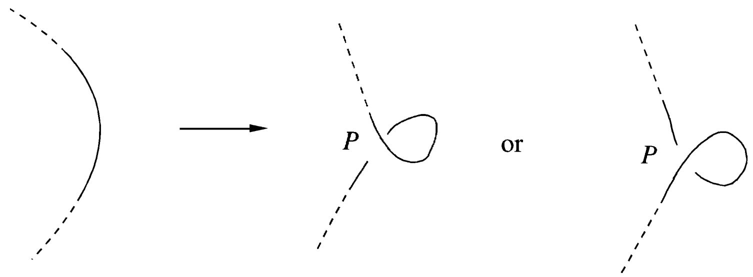

Fig. 2.

3. The invariant W_T for two-dimensional braids in 4-manifolds $F^2 \times \mathbb{R} \times S^1$

Since $\mathrm{Arg}: M^4 \to S^1$ is a trivial fibration, we can identify the fibers $\arg^{-1}(s)$, $s \in S^1$, with $F^2 \times \mathbb{R}$ in a canonical way.

Consequently, the torus $T \hookrightarrow M^4$ can be considered as an isotopy of the knot T_S, $s \in S^1$ fixed, to itself. This is usually called a *motion of a link*. D. Goldsmith was the first to study motions of links in the 3-sphere [7].

Clearly, the knot type of T_S, $s \in S^1$, does not change. For almost all $s \in S^1$ the knot T_S defines a generic diagram with respect to the projection $F^2 \times \mathbb{R} \to F^2$.

Let $\{S_1, \ldots, S_K\} \subset S^1$ be the (finite) set of parameters for which the diagram T_S is not generic but has an *autotangency* (Fig. 3). Autotangencies correspond to Reidemeister moves of type II:

Remark. It is not difficult to see that for a generic two-dimensional braid $T \hookrightarrow M^4$ the other non-generic diagrams T_S have either a triple point or a cusp, corresponding to Reidemeister moves of type III or, respectively, of type I (cf. e.g. [2]).

Let $a \in H_1(F^2; \mathbb{Z}) \backslash \{0, [T_S]\}$ be fixed. For each fixed $s \in S^1 \backslash \{S_1, \ldots, S_K\}$ let $p_s(a)$ denote the (finite) set of all crossings p of the diagram T_S which are of type “a”, i.e. $[(T_s)_p^+] = a$.

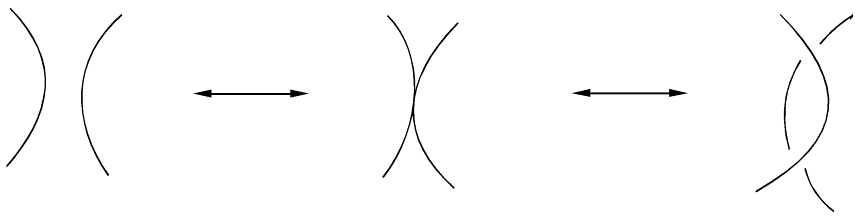

Fig. 3.

Definition 3.1.

$$S_a^o = \bigcup_{s \in S^1 \setminus \{S_1, \dots, S_K\}} p_s(a) \subset F^2.$$

Evidently, for generic two-dimensional braids $T \hookrightarrow M^4$ the set S_a^o is an immersed one-dimensional submanifold of F^2 and the natural projection $S_a^o \to S^1$ given by $p_s(a) \mapsto s$ is locally a diffeomorphism. The writhe $w(p_s(a))$ is constant on each connected component of the immersed submanifold S_a^o.

Definition 3.2. S_a^o is oriented in such a way that the local degree of the projection $S_a^o \to S^1$ is equal to $+1$ if $w(p_s(a)) = +1$ and it is equal to -1 if $w(p_s(a)) = -1$.

Let S_a denote the closure of S_a^o in F^2.

Lemma 3.1. *S_a is a union of oriented immersed circles in F^2.*

Proof. For $s \in S^1 \setminus \{s_1, \dots, s_K\}$ a pair of crossings appear or disappear. These crossings have the same type and opposite writhe. Consequently, the two corresponding oriented arcs in S_a^o become a well oriented arc in S_a. This is illustrated in Fig. 4.

Remark. For $a = 0$ or $[T_S]$ the immersed manifold S_a could have components diffeomorphic to arcs instead of circles (compare the remark after Definition 2.1).

We are now ready to define the invariant W_T.

Definition 3.3.

$$W_T : H_1(F^2; \mathbb{Z}) \setminus \{0, [T_S]\} \to H_1(M^4; \mathbb{Z}) \cong H_1(F^2; \mathbb{Z}) \oplus H_1(S^1; \mathbb{Z})$$

is defined by

$$a \mapsto [S_a] \in H_1(F^2; \mathbb{Z}) \oplus \text{degree } (S_a \to S^1) \in H_1(S^1; \mathbb{Z}).$$

Remark. Evidently, for each $s \in S^1 \setminus \{S^1, \dots, S_K\}$ degree $(S_a \to S^1)|_S = W_{T_s}(a)$.

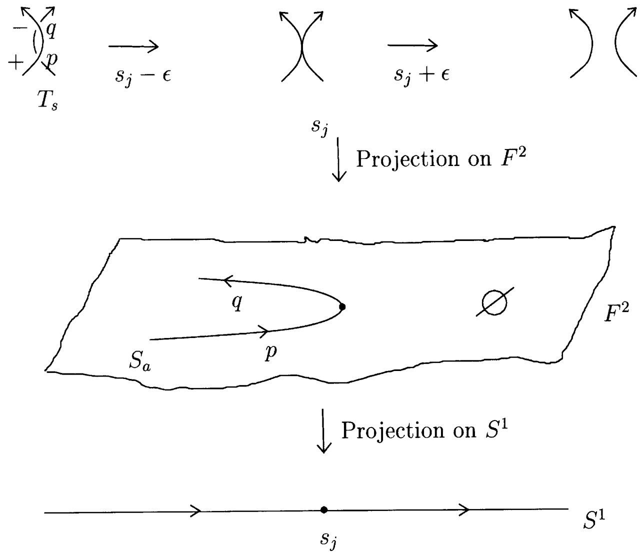

Fig. 4.

4. Invariance of W_T under isotopies consisting of two-dimensional braids

Let $T_t \hookrightarrow M^4$, $t \in [0,1]$, be a braid isotopy, meaning that for each t the map arg stays transversal to T_t. Evidently, neither the knot type $(T_s)_t$, nor the homology class $[(T_s)_t] \in H_1(F^2; \mathbb{Z})$ changes.

$W_{(T_s)_t}$ is a knot invariant and, consequently,

degree$((S_a)_t \to S^1) =$ degree$((S_a)_t \to S^1)|_s = W_{(T_s)_t}(a)$ does not depend on t.

We can assume that $T_t \hookrightarrow M^4$, $t \in [0,1]$, is a generic braid isotopy. Let $\{t_1, \ldots, t_m\} \subset [0,1]$ be a (finite) set of parameters for which the two-dimensional braids T_t is not generic, i.e. there is some $s_i \in S^1$ such that either $(T_{s_i})_{t_i}$ has a singularity of codimension 2 in the space of all diagrams (cf. [5]), or the isotopy $(T_s)_{t_i}$ is for $s = s_i$ tangential to some stratum of singularities of codimension 1 in the space of all diagrams.

We do not need to make a list of all possibilities, because locally the projection of $(S_a)_t$ on S^1 is a diffeomorphism for each fixed t and almost all s. Consequently, $(S_a)_t$ can a priori change only in the following two cases:

(1) $(T_s)_{t_i}$, t_i fixed and $s \in S^1$, passes transversally through a stratum of codimension 2 which consists of an autotangency in a flex.
(2) $(T_s)_{t_i}$, t_i fixed and $s \in S^1$, is ordinarily tangential to a stratum of codimension 1 which consists of an autotangency.

We illustrate both the possibilities in Figs. 5 and 6, where we draw the change of $(T_s)_t$ with respect to s horizontally and with respect to t vertically:

Case 1:

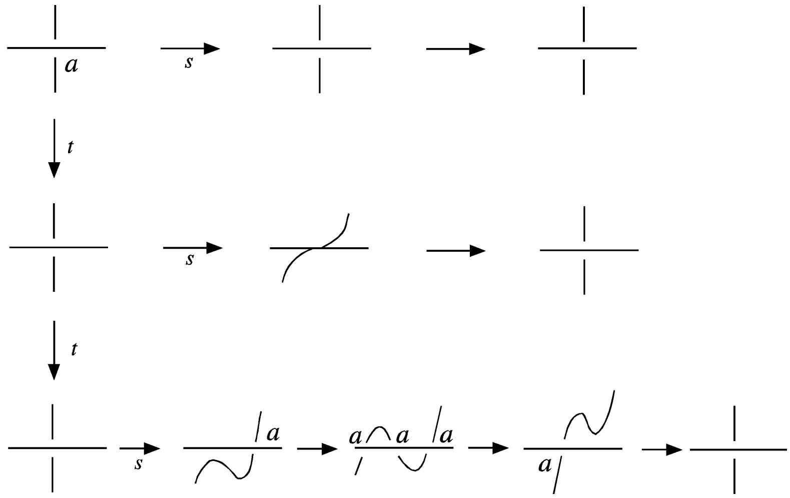

Fig. 5.

Case 2:

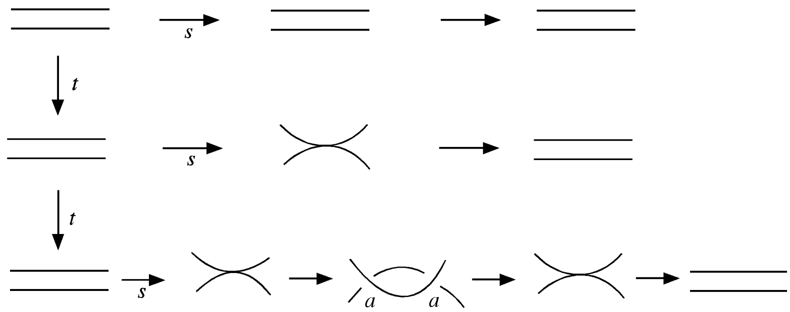

Fig. 6.

or

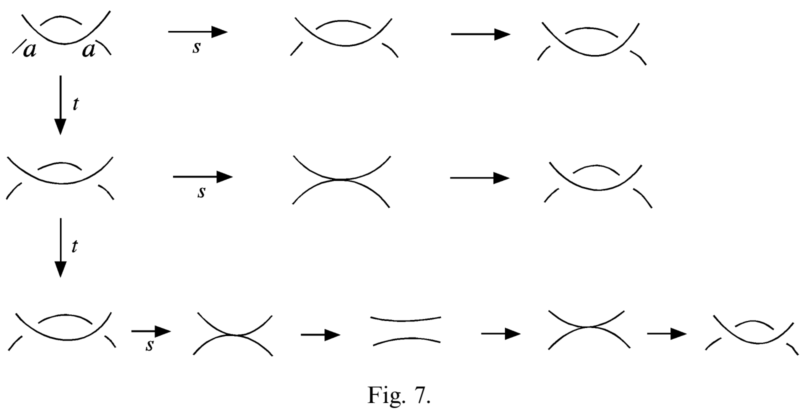

Fig. 7.

We indicated the types of crossings in Fig. 7. In Figs. 8–10, we show the corresponding change of S_a in F^2 (mostly without specifying the orientations on $(T_s)_t$, F^2 the $\mathbb{R}$-factor, the S^1-factor and, hence, on S_a):

Case 1:

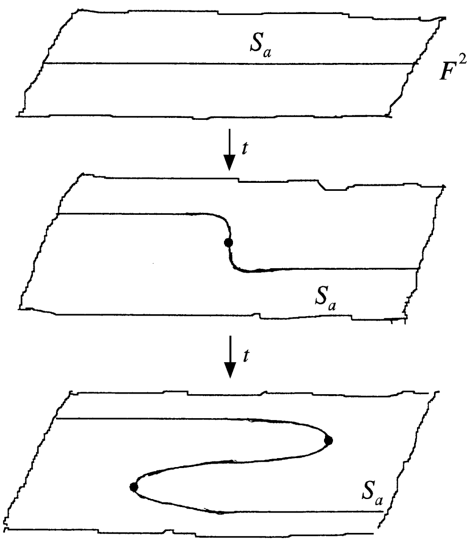

Fig. 8.

Case 2:

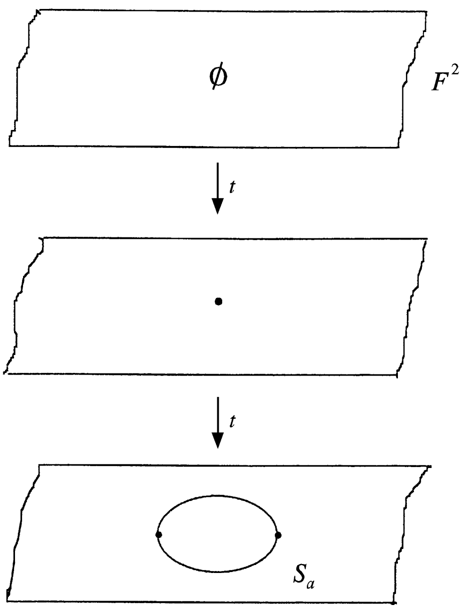

Fig. 9.

or

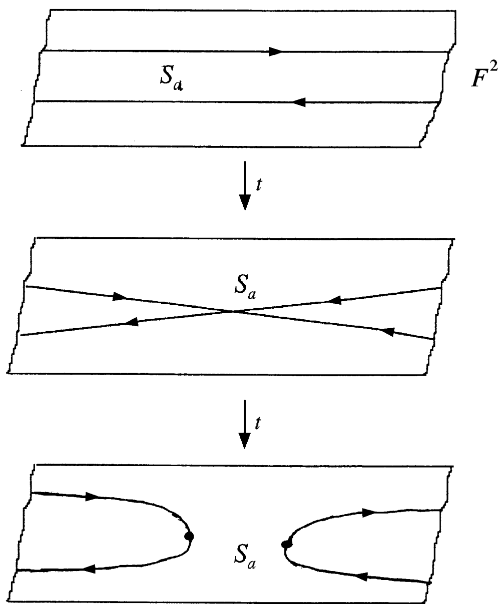

Fig. 10.

In case 1 S_a has not changed at all but only its projection on S^1. In case 2 the components of S_a have actually changed but its homology class $[S_a]$ has not changed. This shows that W_T is indeed invariant under braid isotopies.

5. Invariance of the knot type of the fiber $T_S \hookrightarrow F^2 \times \mathbb{R}$ under smooth isotopies connecting two-dimensional braids

In contrast to the case of classical closed braids for two-dimensional closed braids the knot type $T_S \hookrightarrow F^2 \times \mathbb{R}$ is, of course, not determined by the homology class of $T \hookrightarrow M^4$. Nevertheless we have the following result.

Theorem 5.1. *Let T_t, $t \in [0,1]$, be a smooth isotopy in M^4 of the two-dimensional braid T_0 to the two-dimensional braid T_1. For simplicity we assume that $(T_s)_0$ is connected and not contained in any 3-ball $D^3 \hookrightarrow F^2 \times \mathbb{R}$ for every $s \in S^1$. Then $(T_s)_0$ is isotopic to $(T_s)_1$ in $F^2 \times \mathbb{R} \times \{s\} \cong F^2 \times \mathbb{R}$.*

The proof of the theorem will be a consequence of the following three lemmas.

We consider $\arg_t : T_t \to S^1$ as a 1-parameter family of S^1-valued Morse functions on the embedded torus $T_0 \hookrightarrow M^4$.

Lemma 5.1. *For a generic isotopy T_t, $t \in [0,1]$, there are exactly two types of events which change the isotopy class of the (generic) application $\arg_t : T_t \to S^1$ with fixed t:*

Type 1: an ordinary birth–death singularity

Type 2: a fiber $(T_{s_0})_{t_0}$, $s_0 \in S^1$ fixed, with exactly two singularities of $\arg_{t_0}$.

Proof. Cerf theory [3] says that the events of types 1 and of 2 are the only events which change the isotopy class in a generic abstract 1-parameter family of Morse functions on a torus. In our case the 1-parameter family is induced by embeddings of a torus in a given 4-manifold M^4 together with a given Morse application $M^4 \to S^1$.

But locally, this restricted sort of a 1-parameter family looks the same as the abstract 1-parameter family. Indeed, this follows from the facts that the events of type 1 are local and the events of type 2 are "semi-local", i.e. completely determined by the restriction of $\arg_t$ for t near t_0 on $(T_s)_t$ for s near s_0. The latter is just a tubular neighborhood in the torus of the one-dimensional set $(T_{s_0})_{t_0}$ together with a small deformation of the application $\arg_{t_0}$. It is easy to see that this small deformation on this thickened one-dimensional set can always be induced from the application $M^4 \to S^1$ by an embedding into M^4. □

Remark. We do not say that each Morse application $T \to S^1$ can be globally induced by an embedding of the torus T into a given 4-manifold M^4 together with a given Morse application $M^4 \to S^1$.

We need to fix some notations. Let $(T_{s_0})_{t_0}$ have an ordinary birth–death singularity. Then $(T_s)_t$ looks near $(s_0, t_0) \in S^1 \times [0,1]$, as described in Fig. 11 (up to inversing the directions of s or of t):

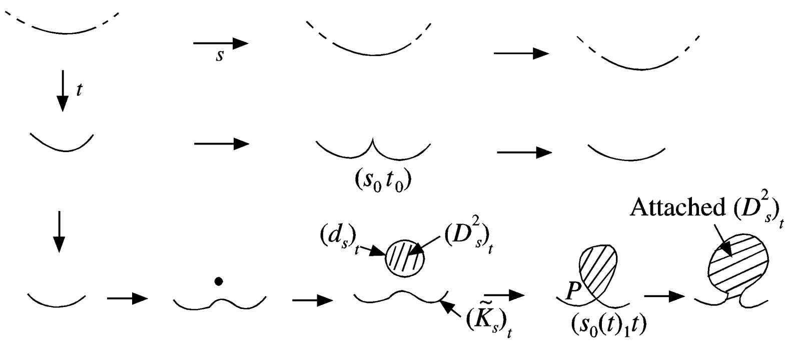

Fig. 11.

We denote by $s_0(t)$ the value of the parameter s for which $(T_{s_0(t)})_t$ has a non-isolated double point (which comes from the ordinary cusp in $(T_{s_0})_{t_0}$). For fixed small $\varepsilon > 0$, $\eta > 0$ the link $(T_{s_0(t)-\varepsilon})_{t_0+\eta} \hookrightarrow F^2 \times \mathbb{R}$ consists of a link $(\tilde{K}_{s_0(t)-\varepsilon})_{t_0+\eta}$, which is isotopic to $(T_{s_0-\varepsilon})_{t_0-\eta}$, and of a trivial knot $(d_{s_0(t)-\varepsilon})_{t_0+\eta}$.

This trivial knot bounds an embedded disk in $F^2 \times \mathbb{R} \backslash (T_{s_0(t)-\varepsilon})_{t_0+\eta}$, which we call $(D^2_{s_0(t)-\varepsilon})_{t_0+\eta}$. In fact, $(d_s)_t$ is a trivial knot because it disappears in the isolated double point and we can choose the disk $(D^2_s)_t$ in such a way that it disappears together with $(d_s)_t$ (cf. Fig. 11).

Let $(K_s)_t$ denote the sublink of $(\tilde{K}_s)_t$ which consists exactly of all non-trivial components

$$(\tilde{K}_s)_t = (K_s)_t \amalg \{\text{trivial link}\}.$$

Here, $\amalg$ denotes the non-connected sum, i.e. the trivial link is contained in the same 3-ball D^3 which does not intersect $(K_s)_t$. (In fact, we will prove that $(K_s)_t$ is a knot isotopic to $(T_s)_o$.)

Lemma 5.2. *Let* $t_{\mathrm{o}} \in [0,1]$ *be such that for almost all* $s \in S^1$ $(T_s)_{t_{\mathrm{o}}} = (K_s)_{t_{\mathrm{o}}} \amalg \{$*trivial link*$\}$, *where* $(K_s)_{t_{\mathrm{o}}}$ *is isotopic to* $(T_s)_{\mathrm{o}}$ *in* $F^2 \times \mathbb{R}$. *If in the isotopy* T_t, $t \in [t_{\mathrm{o}}, t_1]$ *there occur no events of type* 2 *then for almost all* $s \in S^1$ $(T_s)_{t_1} = (K_s)_{t_1} \amalg \{$*trivial link*$\}$, *where* $(K_s)_{t_1}$ *is isotopic to* $(T_s)_{\mathrm{o}}$.

Proof. There are only events of type 1 which we consider with respect to their order by t. Let $(T_{s'})_{t'}$, have an ordinary birth–death singularity. For small $\varepsilon > 0$ the fragment $(T_{s'-\varepsilon})_{t'+\varepsilon} \to (T_{s'})_{t'+\varepsilon} \to (T_{s'+\varepsilon})_{t'+\varepsilon}$ corresponds to performing a band connected sum of $(\tilde{K}_{s'-\varepsilon})_{t'+\varepsilon}$ with $(d_{s'-\varepsilon})_{t'+\varepsilon}$ (cf. Fig. 11).

We have to prove that $(\tilde{K}_{s'-\varepsilon})_{t'+\varepsilon}$ is isotopic to $(T_{s'+\varepsilon})_{t'+\varepsilon}$. This is, evidently, true for t near t_o, because the disk $(D_s^2)_t$ becomes attached to $(T_{s'+\varepsilon})_{t'+\varepsilon}$ and the isotopy from $(T_{s'+\varepsilon})_{t'+\varepsilon}$ to $(\tilde{K}_{s'-\varepsilon})_{t'+\varepsilon}$ can be performed along the (attached) disk $(D_s^2)_t$ (Fig. 11 is realistic for t near t_{o}). But $(T_{s'(t)})_t$ $t \in]t', t_1]$, induces an isotopy of the singular link $(T_{s'(t'+\varepsilon)})_{t'+\varepsilon}$. Indeed, the other events of type 1 can be ignored, because there are no events of type 2 and, hence, the other events of type 1 have no influence on the disk $(D_s^2)_t$ for s near $s_{\mathrm{o}}(t)$. The lemma is proved.

We have to analyze the effect of the events of type 2 on $(T_s)_t$. Let $(T_{s_{\mathrm{o}}})_{t_{\mathrm{o}}}$ be a fiber with exactly two singularities. Evidently, if at least one of the two singularities is an isolated double point then the isotopy type of $(K_s)_t$ cannot change.

Lemma 5.3. *Let the fiber* $(T_{s_{\mathrm{o}}})_{t_{\mathrm{o}}}$ *have two non-isolated double points. We assume that for all* $t < t_0$ *and for almost all* $s \in S^1$ $(T_s)_t = (K_s)_t \amalg \{$*trivial link*$\}$, *where* $(K_s)_t$ *is isotopic to* $(T_s)_0$ *in* $F^2 \times \mathbb{R}$.

Then for small $\varepsilon > 0$ *and for almost all* $s \in S^1$ *the link* $(T_s)_{t_0+\varepsilon} = (K_s)_{t_0+\varepsilon} \amalg \{$*trivial link*$\}$, *where* $(K_s)_{t_0+\varepsilon}$ *is isotopic to* $(T_s)_0$.

Proof. Let p_1 and p_2 denote the double points in $(T_{s_{\mathrm{o}}})_{t_{\mathrm{o}}}$. By abusing notation we denote by p_1 also the corresponding double point in $(T_{s_1(t_0-\varepsilon)})_{t_{\mathrm{o}}-\varepsilon}$ which occurs first with respect to s (in a small arc in S^1 containing s_0) and by p_2 the corresponding double point in $(T_{s_2(t_0-\varepsilon)})_{t_{\mathrm{o}}-\varepsilon}$ (i.e. $s_1(t_0-\varepsilon) < s_2(t_0-\varepsilon)$).

When t passes t_0 then their order changes with respect to s. The assumptions of the lemma imply that there exist attached disks $(D_1^2)_{(s,t)}$ for p_1 and $(D_2^2)_{(s,t)}$ for p_2, such that the isotopy of $(T_s)_t$ first along the disk $(D_1^2)_{(s,t)}$ and then along the disk $(D_2^2)_{(s,t)}$ shows that $(K_s)_t$ has not changed.

We illustrate this in Fig. 12.

We can arrange that for $t := t_0 - \varepsilon$ the disks $(D_1^2)_{(s_1,(t),t)}$ and $(D_2^2)_{(s_1,(t),t)}$ do not intersect. Consequently, we can arrange that, for $s > s_2(t)$ the (attached) disks $(D_1^2)_{(s,t)}$ and $(D_2^2)_{(s,t)}$ either do not intersect at all or they have only ribbon intersections, namely, disjointly embedded arcs connecting the boundary of $(D_2^2)_{(s,t)}$.

We illustrate this in Fig. 13 (cf. also Fig. 12).

Consequently, if the intersection is non-empty, then for $t := t_0 + \varepsilon$ we can a priori no longer use the disks $(D_1^2)_{(s,t)}$ and $(D_2^2)_{(s,t)}$ in order to prove that the type of $(K_s)_t$ has not changed. (The isotopy along $(D_1^2)_{(s,t)}$ would intersect now $(K_s)_t$.) But we can slide the disk $(D_1^2)_{(s,t)}$ over the disk $(D_2^2)_{(s,t)}$ in order to eliminate one after the other the arcs of intersection. This is illustrated in Fig. 14.

The resulting new disk $(\tilde{D}_1^2)_{(s,t)}$ does not intersect $(D_2^2)_{(s,t)}$ any longer and, consequently, the order in which the isotopies of $(T_s)_t$ are performed along the disks does not matter. The lemma is proved. □

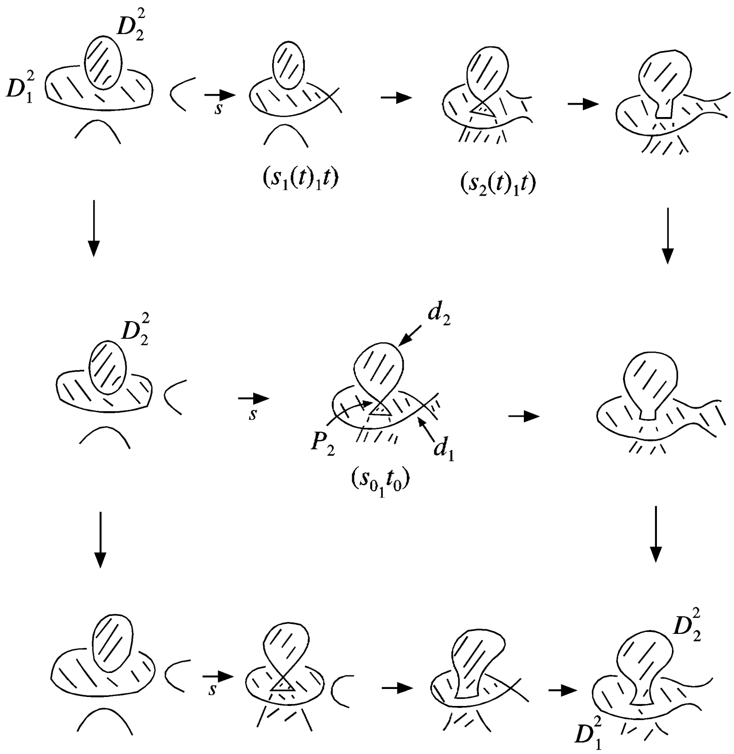

Fig. 12.

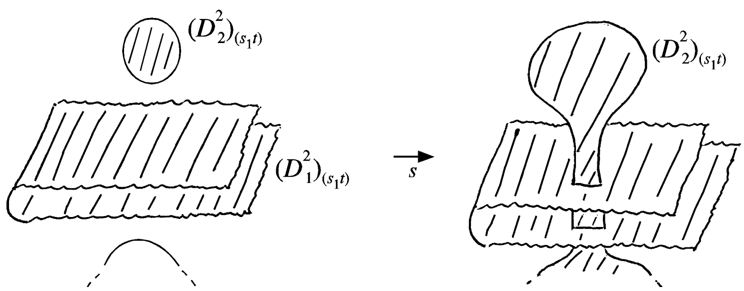

Fig. 13.

Theorem 5.1 now follows from Lemma 5.1 and repeated applications of Lemmas 5.2 and 5.3 for the events of types 1 and 2, respectively, in the isotopy T_t, $t \in [0,1]$. This is possible, because we start from a two-dimensional braid, T_0, and the conclusions of Lemmas 5.2 and 5.3 allow to apply them again for the following events. The theorem is proved. $\square$

Remark. (1) The condition that $(T_s)_0$ is a knot and it is not contained in a 3-ball D^3 is not really necessary for Theorem 5.1 but makes the reduction of the proof simpler. In any case, this

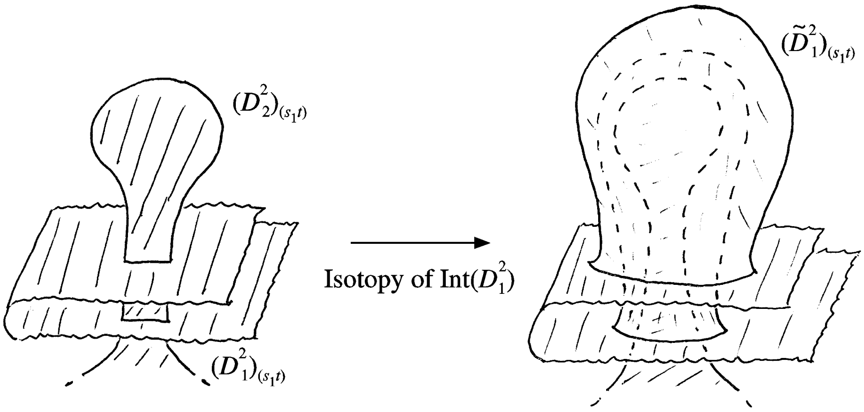

Fig. 14.

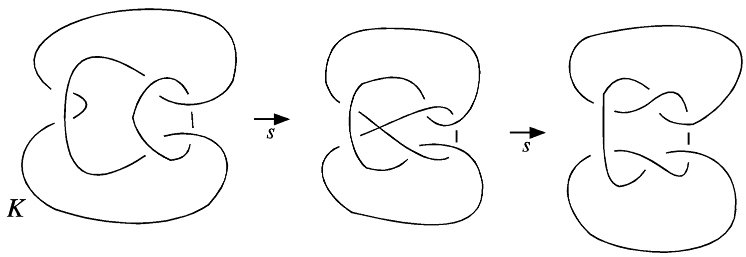

Fig. 15.

condition will be necessary in Theorem 6.1. Moreover, as seen from the proof, the surface F^2 might be non-orientable as well.

(2) A band connected sum of a knot K with a trivial knot is usually called a *fusion*. Fusions were studied in much detail by Kaiser [8]. The following example is, probably, the simplest where a fusion changes the knot type of K.

K is the trivial knot and the result of the fusion is the connected sum of a right trefoil with a left trefoil and, hence, non-trivial (Fig. 15). Theorem 5.1 implies in particular that this fusion cannot be included into a 2-parameter family of links which has the properties described above (an isotopy of a two-dimensional braid).

6. Invariance of W_T under smooth isotopies connecting two-dimensional braids

Let T_t, $t \in [0,1]$, be a smooth isotopy in M^4 of the two-dimensional braid T_0. If for fixed t_0 the torus T_{t_0} is not a two-dimensional braid then the set S_a (used in the definition of W_T) is no longer a cycle. Indeed, the decoration of a crossing p of a link with a homology class $[K_p^+]$ contains non-trivial information only if the two branches which meet at p belong to the same component of the link.

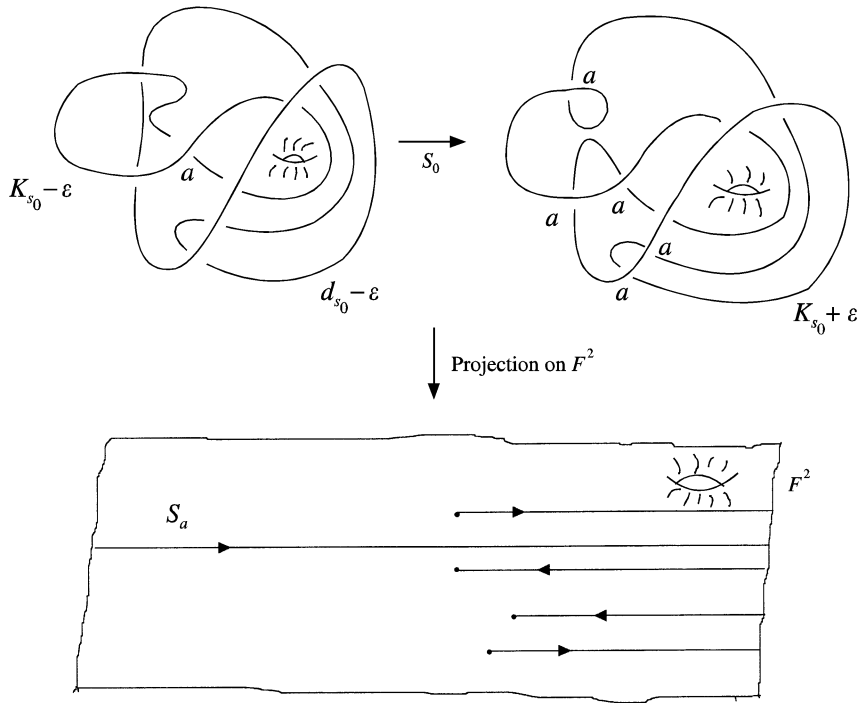

Fig. 16.

Therefore, we could define the invariant W_K only individually for each component of $(T_s)_{t_0}$.

Lemma 5.3 says that for almost all $s \in S^1$ $(T_s)_{t_0} = (K_s)_{t_0} \amalg \{\text{trivial link}\}$, where $(K_s)_{t_0}$ is isotopic to $(T_s)_0$ in $F^2 \times \mathbb{R}$. $W_{\text{trivial knot}} \equiv 0$ and, consequently, we define $W_{(T_s)_{t_0}} := W_{(K_s)_{t_0}}$. The point is now, that if s passes a critical level s_0 where for a small $\varepsilon > 0$ a fusion of a trivial knot $(d_{s_0 - \varepsilon})_{t_0}$ with $(K_{s_0 - \varepsilon})_{t_0}$ happens, then S_a can change in a discontinious way. The reason is, that now crossings of $(d_{s_0 - \varepsilon})_{t_0}$ with itself and of $(d_{s_0 - \varepsilon})_{t_0}$ with $(K_{s_0 - \varepsilon})_{t_0}$ contribute to S_a (defined for $(K_{s_0 + \varepsilon})_{t_0}$). We illustrate this in Fig. 16.

The idea now is to insert for each t_0 into the family $(T_s)_{t_0}$ an isotopy of $(K_{s_0 - \varepsilon})_{t_0}$ to $(K_{s_0 + \varepsilon})_{t_0}$ along the attached disk $(D^2_{s_0 + \varepsilon})_{t_0}$. Let $t_0 \in [0,1]$ be fixed and let $s_0 \in S^1$ be any critical value of $\arg_{t_0}$ such that $(K_{s_0 + \varepsilon})_{t_0}$ is the band connected sum for small $\varepsilon > 0$ of $(K_{s_0 - \varepsilon})_{t_0}$ with a trivial knot $(d_{s_0 - \varepsilon})_{t_0}$. Let $(D^2_{s_0})_{t_0}$ be an attached disk (to the singularity p of $(T_{s_0})_{t_0}$, cf. Lemma 5.2 and Fig. 11).

Let $(d_{s_0})_r$ $r \in [0,1]$, be a differentiable contraction of $d_{s_0} = (d_{s_0})_0$ into $p = (d_{s_0})_1$ and such that for each $r \in [0,1$ $[(d_{s_0})_r$ is a smoothly embedded loop in $(D^2_{s_0})_{t_0}$ which passes through p. We replace the family $(T_s)_{t_0}$, $s \in S^1$, by the following family:

$$(T_s)_{t_0} \quad \text{for } s \in S^1 \backslash s_0,$$

$$(d_{s_0})_{1-r} \cup (T_{s_0} \backslash d_{s_0})_{t_0},\ r \in [0,1] \quad \text{for } s = s_0 \tag{$*$}$$

Family $(*)$ induces a smooth isotopy from $(K_{s_0 - \varepsilon})_{t_0}$ to $(K_{s_0 + \varepsilon})_{t_0}$. We perform this operation for all such $s_0 \in S^1$.

Definition 6.1. $S_a^*(t_0)$ is the oriented one-dimensional set $S_a \subset F^2$ (see Definitions 3.1 and 3.2) associated to family $(*)$.

Lemma 6.1. *$S_a^*(t_0)$ is a cycle for each $a \in H_1(F^2;\mathbb{Z})\backslash\{0,[T_s]_0\}$.*

Proof. This is evident because of the continuity of $p_s(a)$ (cf. Definition 3.1) with respect to s and $1-r$.

Definition 6.2. Let $T_t \hookrightarrow M^4$, $t \in [0,1]$, be a smooth isotopy of tori which starts from a two-dimensional braid T_0, such that $(T_s)_0$ is a knot which is not contained in any 3-ball D^3.

For each fixed $t \in [0,1]$ we define

$$W_{T_t}: H_1(F^2;\mathbb{Z})\backslash\{0,[(T_s)_0]\} \to H_1(F^2;\mathbb{Z}) \oplus H_1(S^1;\mathbb{Z})$$

by $a \mapsto [S_a^*(t)] \in H_1(F^2;\mathbb{Z}) \oplus$ degree $(S_a^*(t) \to S^1) \in H_1(S^1;\mathbb{Z})$

Remark. For t near 0 the definition of W_{T_t} coincides with the definition of W_T for a two-dimensional braid (Definition 3.3).

Theorem 6.1. *Let $T_t \hookrightarrow M^4$, $t \in [0,1]$ be a smooth isotopy from the two-dimensional braid T_0 to the two-dimensional braid T_1. We assume that $(T_s)_o$ is connected and not contained in any 3-ball $D^3 \hookrightarrow F^2 \times \mathbb{R}$. Then $W_{T_t} \equiv W_{T_0}$ for each $t \in [0,1]$ and $W_{T_{t:=1}}$ coincides with the invariant W_T of two-dimensional braids applied to T_1.*

Proof. Theorem 5.1 implies that the knot type of $(K_s)_t$ does not depend on s or t, and, consequently, the knot invariant $W_{(K_s)_t}$ does not depend on s or t neither. But, evidently, $W_{(K_s)_t}(a) = \text{degree}(S_a^*(t) \to S^1)$. So, we have only to prove that $[S_a^*(t)]$ does not depend neither on the chosen contraction $(d_{s_0})_r$, the chosen attached disk $(D^2_{s_0})_{t_0}$ nor the parameter t. This will prove the theorem. $\square$

We do this in a series of lemmas.

We can assume that for fixed t the family $(T_s)_t$, $s \in S^1$, is generic with respect to the projection $F^2 \times \mathbb{R} \to F^2$.

Lemma 6.2. *For fixed $t \in [0,1]$ the class $[S_a^*(t)]$ is independent of the chosen contraction $(d_{s_0})_r$ and of the attached disk $(D^2_{s_0})_t$ up to isotopy of $(D^2_{s_0-\varepsilon})_t$ in $F^2 \times \mathbb{R}\backslash(T_{s_0-\varepsilon}\backslash d_{s_0-\varepsilon})_t$ which fixes $d_{s_0-\varepsilon} = \partial(D^2_{s_0-\varepsilon})_t$.*

(In this formulation we use the fact that by definition the disks $(D^2_{s_0-\varepsilon})_t$ and $(D^2_{s_0})_t$ for small $\varepsilon > 0$ determine each other.)

Proof. $[S_a^*(t)]$ changes continuously (and, hence, is constant) under continuous change of the smooth contraction $(d_{s_0})_r$, $r \in [0,1]$. As well known, the space of smooth contractions of a disk to a fixed point is connected. Consequently, $[S_a^*(t)]$ does not depend on the choice of the contraction $(d_{s_0})_r$ in the disk $(D^2_{s_0})_t$.

$[S_a^*(t)]$ changes also continuously under isotopy of the smooth disk $(D_{s_0}^2)_t$ such that its boundary d_{s_0} is fixed and the interior of the (embedded) disks in the isotopy does not intersect $T_{s_0} \backslash d_{s_0}$. $\square$

The following lemma is of crucial importance. We use the notations of the proof of Lemma 5.3.

Lemma 6.3. *Let $[S_a^*(t_0 - \varepsilon)]$ be defined by using the disks $(D_1^2)_{(s_1,(t),t)}$ and $(D_2^2)_{(s_2,(t),t)}$ for $t := t_0 - \varepsilon$. Let $[S_a^*(t_0 + \varepsilon)]$ be defined by using the disks $(D_2^2)_{(s_2,(t),t)}$ and $(\tilde{D}_1^2)_{(s_1,(t),t)}$ for $t := t_0 + \varepsilon$. Then $[S_a^*(t_0 - \varepsilon)] = [S_a^*(t_0 + \varepsilon)]$.*

Remark. The lemma says that $[S_a^*(t)]$ depends only on the isotopy type of the attached disk $(D_{s_0}^2)_t$ in $F^2 \times \mathbb{R} \backslash (K_{s_0})_t$ even if the trivial components of $(T_{s_0})_t$ are fusioned with $(K_{s_0})_t$.

Proof. $(T_{s_0})_{t_0}$ has exactly two non-isolated double points. $(T_{s_0})_{t_0} = K \cup d_1 \cup d_2 \amalg \{\text{trivial link}\}$, where K is isotopic to $(T_s)_0$, and d_1 and d_2 are trivial knots attached to K in the double points p_1 and p_2 (cf. Fig. 12).

For small $\eta > 0$ there appear five new sorts of crossings in $(K_{s_0+\eta})_{t_0}$ with respect to $(K_{s_0-\eta})_{t_0}$:

(1) Crossings with both branches in d_1.
(2) Crossings with both branches in d_2.
(3) Crossings with one branch in K and one in d_1.
(4) Crossings with one branch in K and one in d_2.
(5) Crossings with one branch in d_1 and one in d_2.

We have two ways to eliminate the new crossings:
(A) by contraction in the disks D_1^2 and D_2^2 for $t := t_0 - \varepsilon$.
(B) by contraction in the disks $\tilde{D}_1^2$ and D_2^2 for $t := t_0 + \varepsilon$ (cf. Figs. 12 and 14).

Crossings of type (1) or (3) are eliminated in (A) by D_1^2 and in (B) by $\tilde{D}_1^2$. But D_1^2 is isotopic to $\tilde{D}_1^2$ in $F^2 \times \mathbb{R} \backslash K$ and, consequently, the crossings of type (1) as well as those of type (3) give the same contributions to $[S_a^*(t)]$ in both cases (A) and (B).

Crossings of type (2) or (4) are eliminated in (A) and (B) the same way and, hence, give also the same contributions to $[S_a^*(t)]$.

In case (A) the crossings of d_1 with d_2 are eliminated in two different ways: by the contraction $(d_1)_r$ in D_1^2 and by the contraction $(d_2)_r$ in D_2^2.

Composing $(d_1)_r$, $r \in [0,1]$, with $(d_2)_{1-r}$, $r \in [0,1]$, gives for each $a \in H_1(F^2;\mathbb{Z}) \backslash \{0,[T_S]_0\}$ a cycle, which we call S_a. In case (B) we compose $(d_1)_r$ with $(\tilde{d}_2)_{1-r}$, $r \in [0,1]$, where $(\tilde{d}_2)_r$ is the contraction in $\tilde{D}_2^2$. We call the corresponding cycles $\tilde{S}_a$. We have to prove that $[S_a] = [\tilde{S}_a]$.

But this follows from $[S_a] = [\tilde{S}_a] = 0$. Indeed, we consider, e.g. the 1-parameter family of cycles S_a^z, $z \in [0,1]$, induced by the composition of $(d_1)_r$, $r \in [0,1]$, with $(d_2)_{1-r}$, $r \in [0,z]$. Evidently, the class $[S_a^z]$ changes continuously with z and, hence, is constant. But S_a^z, $z \in [0,1]$ connects S_a with the empty cycle ($(d_2)_0$ is a point). The lemma is proved. $\square$

For the next lemma we use the same notations as in Lemma 6.2.

Lemma 6.4. *The disk $(D_{s_0-\varepsilon}^2)_t$ is unique up to isotopy in $F^2 \times \mathbb{R} \backslash (K_{s_0-\varepsilon})_t$.*

Proof. $\pi_2(F^2 \times \mathbb{R} \backslash (K_{s_0-\varepsilon})_t) = 0$, because $(K_{s_0-\varepsilon})_t$ is connected and not included in any 3-ball $D^3 \hookrightarrow F^2 \times \mathbb{R}$. It follows then by standard cut and past arguments (see e.g. [10]) that any two embedded disks in $F^2 \times \mathbb{R} \backslash (K_{s_0-\varepsilon})_t$ with the same boundary are isotopic with fixed boundary in $F^2 \times \mathbb{R} \backslash (K_{s_0-\varepsilon})_t$. $\square$

Lemmas 6.2–6.4 imply that for fixed t the invariant W_{T_t} is well defined and that for a two-dimensional braid the invariant W_{T_t} coincides with the previously defined invariant W_T.

Lemma 6.5. *W_{T_t} does not depend on t.*

Proof. In the proof of Lemmas 6.2 and 6.3 we have assumed that for fixed t the family $(T_s)_t$, $s \in S^1$, is generic with respect to the projection $F^2 \times \mathbb{R} \to F^2$. Of course, we can no longer assume this if t varies in a 1-parameter family. Let p be a non-isolated double point in $(T_{s_0})_{t_0}$.

There are exactly two generic events:

(1) The plane spaned by the tangent vectors to $(T_{s_0})_{t_0}$ at p contains the direction of the projection $F^2 \times \mathbb{R} \to F^2$.

(2) The singularity p is in the preimage of a crossing.

In order to show that $[S_a^*(t)]$ does not change we compare the pictures for $(s_0(t_0 - \varepsilon), t_0 - \varepsilon)$ and $(s_0(t_0 + \varepsilon), t_0 + \varepsilon)$.

We illustrate case (1) in Fig. 17.

There appear two new crossings (of different writhe). But because d is a trivial knot and $[K] = [(T_s)_0]$ one easily sees that the two new crossings are marked by $[(T_s)_0]$ or by 0 and, hence, do not contribute to any $S_a^*(t)$.

In case (2) a branch of d or of K goes over p. By examining all the possibilities one easily sees that $[S_a^*(t)]$ does not change. We illustrate this just in one example in Fig. 18.

The two crossings which go over p have the same markings, say "a", if d is attached to K, and they have different writhes.

The corresponding pieces of $S_a^*(t_0 - \varepsilon)$ and of $S_a^*(t_0 + \varepsilon)$ look alike in Fig. 19.

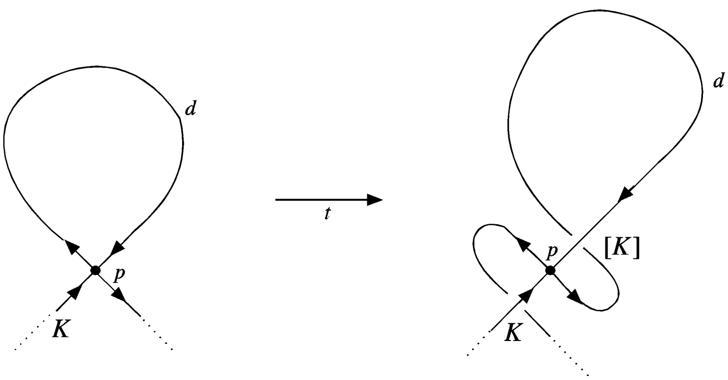

Fig. 17.

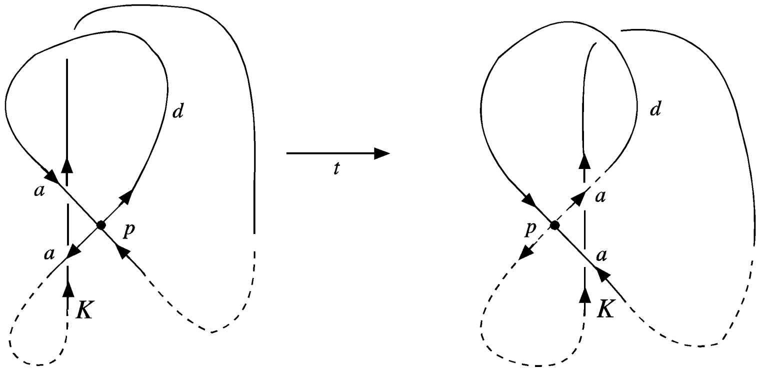

Fig. 18.

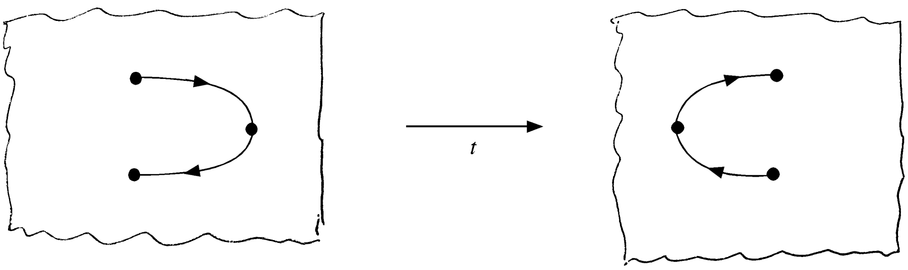

Fig. 19.

Consequently, $[S_a^*(t)]$ has not changed. Before d is attached the pieces of $S_a^*(t_0 - \varepsilon)$ and $S_a^*(t_0 + \varepsilon)$ look exactly the same. Besides the events of non-generic projections, we have to consider the critical values of t. Evidently, $[S_a^*(t)]$ is invariant if t passes a critical value corresponding to a birth–death singularity, because there are no new crossings at all.

Lemma 6.3 implies that $[S_a^*(t)]$ is invariant if t passes a critical value corresponding to a couple of singularities in the same fiber $(T_{s_0})_{t_0}$. The lemma and, hence, the theorem is proved.

Remark. (1) For the invariance of W_{T_t} it is of crucial importance that the torus T_t is included in a family which starts from a two-dimensional braid. Assume that $T_{t_1} \hookrightarrow M^4$ is a torus such that for almost all $s \in S^1$ $T_s = K_s \amalg \{\text{trivial link}\}$, where the knot type K_s is fixed. Then we could define $W_{T_{t_1}}$, but it would not be an isotopy invariant. The reason is that in an isotopy to another such torus T_{t_2}, even with the same knot type K_s, there could appear band connected sums which change the knot type of K_s. In this case W_{T_t} would no longer be defined and there is no possibility to relate W_{T_1} to W_{T_2}.

(2) The invariant W_{T_t} can be refined to an invariant

$$W_{T_t}\colon \{\text{free homotopy classes of oriented loops in } F^2\} \backslash \{0, \text{class of } (T_s)_o\}$$
$$\to H_1(F^2;\mathbb{Z}) \oplus H_1(S^1;\mathbb{Z})$$

by replacing the homological marking of a crossing p by the free homotopy class of K_p^+.

7. Applications

We consider only very simple examples with $F^2 := \mathbb{R}^2\backslash 0$. We draw the axes $0\times\mathbb{R}\hookrightarrow\mathbb{R}^2\times\mathbb{R}$ as a point on the plane. All our two-dimensional braids in the examples will be constructed in the following way: we fix an oriented diagram T_s of a knot in $(\mathbb{R}^2\backslash 0)\times\mathbb{R}$. The orientation-preserving diffeomorphism of M^4, which is induced by identifying $(\mathbb{R}^2\backslash 0)\times\mathbb{R}\times\{0\}$ with $(\mathbb{R}^2\backslash 0)\times\mathbb{R}\times\{1\}$ by a rotation of the angle $2\pi n$, $n\in\mathbb{Z}$, around the axes $0\times\mathbb{R}$, is called φ_n. For positive n the rotation is assumed to be in counterclockwise direction. The two-dimensional braid $T(n,T_s)\hookrightarrow M^4$ is defined as $\varphi_n(T_s\times S^1)$.

Let m denote the generator of $H_1(\mathbb{R}^2\backslash 0;\mathbb{Z})\cong\mathbb{Z}$ which is represented by the positive meridian of the axes $0\times\mathbb{R}$: $\odot^{m}$.

Example 1. (see Fig. 20). T_s is the trivial knot in $\mathbb{R}^2\times\mathbb{R}$.

One easily calculates

$$W_{T(n,T_s)} = \begin{cases} -m\mapsto -nm\oplus -1, \\ 2m\mapsto -nm\oplus -1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Consequently, Theorem 6.1 *implies that all the $T(n,T_s)$ are pairwise not smoothly isotopic in M^4.*

However, by construction, all pairs $(M^4, T(n,T_s))$ are pairwise diffeomorphic. In particular, all the two-dimensional knots $T(n,T_s)$ have the same complement in M^4. Moreover, T_s is regularly homotopic in $(\mathbb{R}^2\backslash 0)\times\mathbb{R}\times\{s\}$ to the positive meridian of $0\times\mathbb{R}\times\{s\}$. Performing this homotopy in all fibers $(\mathbb{R}^2\backslash 0)\times\mathbb{R}\times\{s\}$ simultaneously, we see that all $T(n,T_s)$ are regularly homotopic in $M^4=S^1\times S^1\times\mathbb{R}^2$ to the 0-section $S^1\times S^1\times 0$.

I do not know, whether or not there are $T(n,T_s)$ which are topologically isotopic.

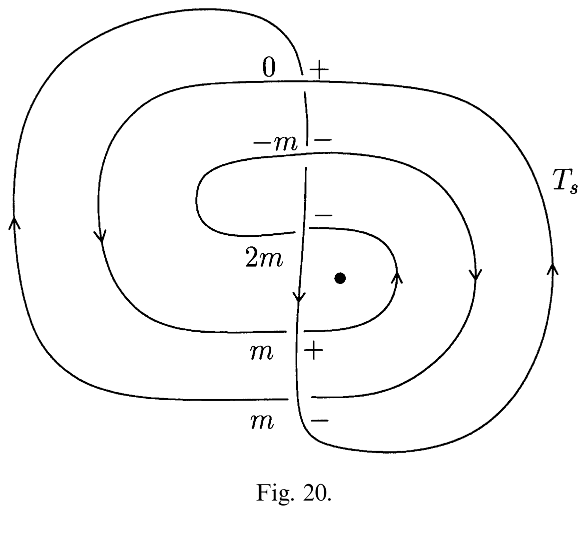

Fig. 20.

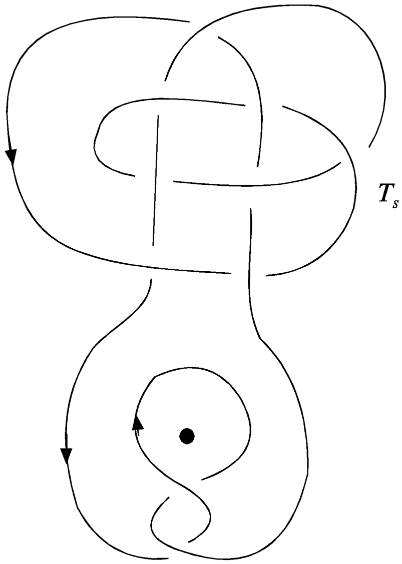

Fig. 21.

Example 2. (see Fig. 21). T_s is the knot 8_{17} in $\mathbb{R}^2 \times \mathbb{R}$ (see [10]). It is well known (see [2]) that T_s is not isotopic to its inverse $-T_s$. Evidently, $T(n, -T_s) = -T(n, T_s)$ and $T(n, T_s)$ is regularly homotopic to $-T(n, T_s)$ for each fixed n (cf. Example 1).

Theorem 5.1 implies that $T(n, T_s)$ is not smoothly invertible (i.e. smoothly isotopic to $-T(n, T_s)$) for each fixed n. Is $T(n, T_s)$ topologically invertible?

Remark. $T(n, T_s)$ in Example 1 represents a primitive class in $H_2(M^4; \mathbb{Z})$ and, hence, there are no coverings over M^4 ramified along $T(n, T_s)$.

The diffeomorphism type of the total space of a ramified covering does not depend on the orientation of the ramification set. Hence, one cannot distinguish $T(n, T_s)$ from $-T(n, T_s)$ in Example 2 by studying ramified coverings.

Consequently, one cannot apply the method from [6] in order to distinguish in our examples the two-dimensional knots up to smooth isotopy.

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